

PEIRCE THE LOGICIAN

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SUMMARIES

This paper traces the influence of the Boolean school, and more specifically of Peirce and his students, on the development of modern logic. In the 1890s it was Schröder's Algebra der Logik that represented the state of the art. This work mentions Frege, but the quantifier notation it adopts (a variant of the modern notation) is credited to Peirce and his students O. H. Mitchell and Christine Ladd-Franklin. This notation was widely adopted; both Zermelo and Löwenheim wrote famous papers in Peirce-Schröder notation. Even Whitehead (in 1908, in his Universal Algebra) fails to mention Frege, but cites the "suggestive papers" by Mitchell and Ladd-Franklin. (Russell credits Frege, with many things, but nowhere credits him with the quantifier; if the quantifiers in Principia were devised by Whitehead, they probably come from Peirce). The aim of this paper is not to detract from our appreciation of Frege's great work, but to emphasize that its influence came largely after 1900 (after Russell pointed out its significance). Although Frege discovered the quantifier in 1879 and Peirce's student Mitchell independently discovered it only in 1883, it was Mitchell's discovery (as modified and disseminated by Peirce) that made the quantifier part of logic. And neither Löwenheim's theorem nor Zermelo set-theory depended on Frege's work at all, but only on the work of the Boole-Peirce school.

Nous retraçons, dans cet article, l'influence sur le développement de la logique moderne de l'école booléenne, et plus spécifiquement de Peirce et de ses étudiants. Au cours de la dernière décennie du XIX^e siècle, le livre de Schröder Algebra der Logik constituait un summum dans son genre. On y mentionnait Frege, mais la notation des quantificateurs adoptée (une variante de la notation moderne) est attribuée à Peirce et à ses étudiants O. H. Mitchell et Christine

Ladd-Franklin. L'emploi de cette notation fut largement répandue. Zermolo et Löwenheim l'utilisèrent dans des articles célèbres. Même Whitehead (en 1908, dans son *Universal Algebra*) ne mentionne pas Frege; cependant, il se réfère à l'"article évocateur" de Mitchell et Ladd-Franklin. (Russell attribue beaucoup de choses à Frege, toutefois à aucun moment il ne le fait pour les quantificateurs. Si les quantificateurs des *Principia* furent mis au point par Whitehead, ils originent probablement de Peirce.) Le but de cet article n'est pas de rabaisser l'évaluation de l'oeuvre si fondamentale de Frege mais plutôt de faire ressortir que son influence est en grande partie postérieure à 1900 (après que Russell en fit remarquer l'importance). Bien que Frege ait découvert le quantificateur en 1879 et que Mitchell, l'étudiant de Peirce, ne l'ait découvert indépendamment qu'en 1883, ce fut par la découverte de Mitchell (telle que modifiée et répandue par Peirce) que le quantificateur prit place en logique. Ni le théorème de Löwenheim, ni la théorie des ensembles de Zermolo ne reposèrent sur l'oeuvre de Frege. Ils s'édifièrent uniquement sur l'oeuvre de Boole et Peirce.

In diesem Beitrag wird der Einfluß der Boole'schen Schule, insbesondere derjenige von Peirce und seinen Schülern, auf die Entwicklung der modernen Logik untersucht. In den 90er Jahren des 19. Jahrhunderts stellte Schröders "Algebra der Logik" den damals erreichten Stand dar. Darin wird Frege erwähnt, aber das verwendete Zeichen für den Quantor (eine Variante der modernen Symbolik) wird Peirce und seinen Schülern O. H. Mitchell und Christine Ladd-Franklin zugeschrieben. Diese Symbolik wurde weithin übernommen; sowohl Zermelo wie Lowenheim schrieben berühmte Abhandlungen in der Bezeichnungsweise von Peirce-Schröder. Selbst Whitehead erwähnt in der "*Universal Algebra*" (1908) Frege nicht, zitiert aber die "vorbildlichen Abhandlungen" von Mitchell und Ladd-Franklin. (Russell schreibt Frege vieles zu, doch nirgends den Quantor; falls die Quantoren in den "*Principia*" von Whitehead entwickelt wurden, gehen sie wahrscheinlich auf Frege zurück.) Es ist nicht Absicht dieses Aufsatzes, unsere Wertschätzung von Freges bedeutendem Werk zu verkleinern, sondern zu betonen, daß sein Einfluß im wesentlichen nach 1900 zu verzeichnen ist (nachdem Russell seine Bedeutung hervorgehoben hatte). Obgleich Frege den Quantor 1879 und Peirces Schüler Mitchell ihn

unabhängig 1883 entdeckt hatten, wurde durch Mitchells Entdeckung (in der von Peirce modifizierten und verbreiteten Form) der Quantor Bestandteil der Logik. Weder Lowenheims Theorem noch die Mengenlehre Zermelos waren von Freges Untersuchungen abhängig, vielmehr stützten sich beide auf Arbeiten der Bocle-Peirce-Schule.

My topic is hardly one on which I can be said to have specialized knowledge or training. I am not a historian of logic, nor (although I regard Peirce as a towering giant among American philosophers) am I a "Peirce scholar." Indeed, it is highly unlikely that I shall tell you anything that Professor Eisele could not tell you better. But I am a working logician who found himself at one time led, if only for a few months and if only out of a quite personal curiosity, to research the early history of mathematical logic, and thereby to discover just what Peirce's contribution to and stature in 19th-century logic was; and it may be fitting if I use this occasion to "debrief" myself, as the intelligence agencies say.

I shall not concern myself with the actual details of Peirce's system, although there are matters there that are of interest to students of Peirce's philosophy. (For example, Hans Herzberger has verified a claim that Peirce often makes, that in the Peircian "logic of relatives" all four- and higher-term relations are reducible to triadic relations, but that it is not possible in general to reduce a higher-degree relation to diadic relations. This claim has metaphysical significance within Peirce's system because of its connection with "thirdness" [Herzberger 1981].) Nor shall I concern myself with what is or has become esoteric in Peirce's logical work, as the method of existential graphs has become. Rather, my aim is to tell you how much that is quite familiar in modern logic actually became known to the logical world through the efforts of Peirce and his students.

MY "BOOLEAN" MOTIVATION

What triggered my investigations was a certain admiration I have for George Boole, and a certain piece of disrespect to Boole and his followers on the part of Quine. I would not have fully appreciated Boole's mathematical mind if it had not been for a couple of accidents, which I must tell you about. In my twenties I knew, of course, that Boole was the inventor of "Boolean algebra" (which I knew from contemporary texts, not from 19th-century ones). I had learned that he had taken holy orders, and not knowing then what I know now about early-19th-century England, I pictured Boole much more as a clergyman than

as a mathematician. I supposed he might have been an amateur whose discovery of mathematical logic was rather the product of one good idea and some modest mathematical ingenuity than of real power.

Just as I turned thirty, however, I was working on Hilbert's 10th Problem with Martin Davis. In the course of a summer's research he and I (with some crucial help from Julia Robinson) succeeded in showing the unsolvability of the decision problem for exponential Diophantine equations (a result which Matyasevich improved a few years later to complete the "negative solution" of Hilbert's 10th Problem--that is, the proof that the decision problem for ordinary Diophantine equations is unsolvable.) The fact is that the first proof of our result that Davis and I found used a bit of complicated mathematical analysis. This was later eliminated when Julia Robinson simplified the entire proof drastically and eliminated one unproved number-theoretic hypothesis that Davis and I had had to use. We had to use, in fact, the theory of the gamma function, which we learned from Whittaker and Watson and an old mathematics text--Boole's *Calculus of Finite Differences* [1860], a theory of the evaluation of indefinite sums and products that Boole developed on the basis of an ingenious generalization of the ideas of the differential and integral calculus. At this point I became aware that Boole was not a clerical amateur but a very high powered mathematician.

During the summer Davis taught me a little bit of modern "operator methods" in differential equations. These methods, which are based on the idea of a "ring of operators," go back directly to another idea of Boole's [1859]--the idea of treating the symbol for differentiation as if it were the name of a strange kind of number. Indeed, Boole solved differential equations by exactly the method that was just then (in the late fifties) beginning to work its way down into the more sophisticated introductory texts.

This made me interested in Boole in earnest, and a little reading soon convinced me that Boole had a remarkable mathematical program (which he shared with a certain school of British analysts) and that his discovery of mathematical logic was the direct result of that program. The program--the program of the "Symbolic School" of British analysts--is today out of date, but in a certain sense it was the bridge between traditional analysis (real and complex analysis) and modern abstract algebra. In effect, Boole and his co-workers were struggling for the notion of an abstract ring, in the modern mathematical sense, i.e., a structure with an addition and a multiplication. They could not quite come up with this idea, but they did see that they wanted to free the methods of algebra from an exclusive concern with their traditional content, the real and complex numbers. So they started to use algebraic methods beyond

their ability to give those methods an interpretation. I do not mean to use these terms confusedly or vaguely, but rather formally: in fact, Boole was quite conscious of the idea of *disinterpretation*, of the idea of using a mathematical system as an algorithm, transforming the signs purely mechanically without any reliance on meanings. In connection with logic, this very important idea appears on the opening pages of Boole's *Laws of thought* [1854].

Boole's work, then, was an extension of algebra beyond its previous bounds; the algebra of classes, the calculus of finite differences, and the contributions to the theory of differential equations were simply three parts of what Boole (and the Symbolic School generally) saw as a unified enterprise. Modern mathematical logic is something the early history of which cannot be separated from the development of modern mathematics in the direction of abstractness.

Even before I came to appreciate Boole's "power" as a mathematician, I had known that prior to Boole logic had been in a straightjacket of sorts because of the fossilization of Aristotelean logic during the late middle ages. Almost all logic texts prior to Boole (and many after Boole) taught nothing but syllogistic logic, or syllogistic logic with insignificant extensions and improvements. All premises, in the inferences discussed, had exactly two "terms," or class names, in them: this was because only premises in the old "categorical forms" (*A*, or "All *S* are *P*," *I*, or "Some *S* are *P*," *E*, or "no *S* are *P*," and *O*, or "Some *S* are not *P*") were considered. After Boole introduced mathematical notation (as Leibniz had done earlier, but without issue) inferences could be studied which contained any number of terms and the premises and conclusions of which contained as many "ors," "ands," and "nots" (or class unions, class intersections, and complementations--Boole gave both interpretations) as you please.

It is true that the system was still very weak, compared to modern quantification theory. Dyadic and higher-degree relations could not be symbolized, and there were no quantifiers. But the search for an improvement in the first respect, a logic or "algebra" of relations, began at once, and the quantifiers were invented thirty-two years after the appearance of Boole's *The Mathematical Analysis of Logic* [1847] (in Frege [1879]; also by Peirce's student O. H. Mitchell [1883]). I assumed that everyone realized that with the appearance of a complete "algebra of classes" the dam was broken, and (given the mathematical sophistication of the age) the subsequent development was inevitable. It seemed inconceivable to me that anyone could date the continuous effective development of modern mathematical logic from any point other than the appearance of Boole's two major logical works, the *Mathematical Analysis* and the *Laws of Thought*.

Yet, in a very widely used text by a philosopher I admire enormously, I read that "logic is an old subject and since 1879 it has been a great one" [Quine 1950]. In short, logic only broke out of its long stagnation with Frege. In one pen stroke Quine dismisses the entire Boolean school--of which Peirce was, in a sense, the last and greatest figure. In van Heijenoort's *Source Book* [1967] I detect a similar bias (though not as extremely stated): the Booleans are mentioned only for the purpose of unflattering comparison with Frege and later authors. It is this biased account, the "logic was invented by Frege" account, that I want to rebut today. In the process, I can do honor to Peirce as well as to Boole--and honoring one's intellectual heroes is one of the purest and most self-sufficient of life's pleasures.

THE BIRTH OF THE QUANTIFIER

I do not know exactly what Quine meant by his remark; I suspect it was meant simply as an appreciation of the great richness and power of the two systems constructed by Frege (*Begriffsschrift* was very similar--except for the unattractive notation Frege used--to today's standard systems of second-order logic; the *Grundgesetze* system, which Russell proved inconsistent, sketched the whole program of developing modern mathematics within higher predicate calculus, the program Russell repaired by inventing the theory of types). But to many readers of Quine's *Methods of Logic* it has come to mean something quite different. The impression which has unfortunately become widespread is that Frege discovered the quantifier not only in the sense of discovering it four years earlier than O. H. Mitchell (Peirce's student and co-worker), but in the sense that it was Frege who, so to speak, launched the whole mighty ship by himself. Reading the Heijenoort *Source Book* does little to dispel this impression. Certainly I must have believed something like this; otherwise I would not have been so surprised to discover the facts which follow.

When I started to trace the later development of logic, the first thing I did was to look at Schröder's *Vorlesungen über die Algebra der Logik* [1890-1895]. This book, which appeared in three volumes, has a third volume on the logic of relations (*Algebra und Logik der Relative*, 1895). The three volumes were the best-known logic text in the world among advanced students, and can safely be taken to represent what any mathematician interested in the study of logic would have had to know, or at least become acquainted with, in the 1890s.

As the title suggests, the approach was algebraic (Boole's logic, as we saw, grew out of abstract algebra), and the great

problem was to develop a logic of *Relative* (i.e., relations). (The influence of the German word *Relativ* is, perhaps, the reason Peirce always wrote "relatives" and not "relations.") Peirce, although himself a member of the algebraic school (he criticized himself for this, in his correspondence) had reservations about Schröder's close assimilation of logical problems to algebraic ones. "While I am not at all disposed to deny that the so called 'solution problems', consisting in the ascertainment of the general forms of relatives which satisfy given conditions, are often of considerable importance, I cannot admit that the interest of logical study centers in them," Peirce writes [Peirce 1931, 3.512]. And "Since Professor Schröder carries his algebraicity so very far, and talks of 'roots', 'values', 'solutions', etc., when, even in my opinion, with my bias towards algebra, such phrases are out of place ..." [Peirce 1931, 3.519]. But my purpose in consulting this reference work was narrower; I simply wished to see how Schröder presented the quantifier.

Well, Schröder does *mention* Frege's discovery, though just barely. But he does not *explain* Frege's notation at all. The notation he both explains and adopts (with credit to Peirce and his students, O. H. Mitchell and Christine Ladd-Franklin) is Peirce's. And this is no accident: Frege's notation (like one of Peirce's schemes, the system of existential graphs) repelled everyone (although Whitehead and Russell were to study it with consequential results). Peirce's notation, in contrast, was a typographical variant of the notation we use today. Like modern notation it lends itself to writing formulas on a line (Frege's notation is two-dimensional) and to a simple analysis of normal form formulas into a prefix (which Peirce calls the Quantifier) and a matrix (which Peirce calls the "Boolean part" of the formula).

Moreover, as Warren Goldfarb [1979] has emphasized in a fine paper on the history of the quantifier, the Boolean school, including Peirce, was willing to apply logical formulas to different "universes of discourse," and Peirce was willing (unlike Frege) to treat first-order logic by itself, and not just as part of an ideal language (with a fixed universe of discourse, viz., "all objects," for Frege). In fact--and this may surprise you as it surprised me--the term "first order logic" is due to Peirce! (It has nothing to do with either Russell's theory of types or Russell's theory of *orders*, although the way Peirce distinguished between first-order and second-order formulas--by whether the "relative" is quantified over or not--obviously has something to do with logical type.) In summary, Frege tried to "sell" a grand logical-metaphysical scheme with a dubious ontology, while Peirce (and, following him, Schröder) was busy "selling" a modest, flexible, and extremely useful notation.

The success they experienced was impressive. While, to my knowledge, no one except Frege ever published a single paper in Frege's notation, many famous logicians adopted Peirce-Schröder notation and famous results and systems were published in this notation. Löwenheim stated and proved the Löwenheim theorem (later reproved and strengthened by Skolem, whose name became attached to it together with Löwenheim's) in Peircian notation. In fact, there is no reference in Löwenheim's paper to any logic other than Peirce's. To cite another example, Zermelo presented his axioms for set theory in Peirce-Schröder notation, and not, as one might have expected, in Russell-Whitehead notation.

One can sum up these simple facts (which anyone can quickly verify--I warned you that I am no historian) as follows: Frege certainly discovered the quantifier *first* (four years before O. H. Mitchell, going by publication dates, which are all we have as far as I know). But Leif Erikson probably discovered America "first" (forgive me for not counting the native Americans who of course *really* discovered it "first"). If the effective discoverer, from a European point of view, is Christopher Columbus, that is because he discovered it so that it *stayed* discovered (by Europeans, that is), so that the discovery became *known* (by Europeans). Frege did "discover" the quantifier in the sense of having the rightful claim to priority. But Peirce and his students discovered it in the effective sense. The fact is that until Russell appreciated what he had done, Frege was relatively obscure and it was Peirce who seems to have been known to the entire world logical community. How many of the people who think that "Frege invented logic" are aware of these facts?

The example of Löwenheim shows something else: *metamathematical* work (of a certain kind) did not have to wait for Russell and Whitehead to make Frege's work known (and to extend it and repair it). First-order logic (and its metamathematical study) would have existed without Frege. (Zermelo even denied that his *set-theoretic* work depended on Whitehead and Russell; he claimed to have been aware of the "Russell paradox" on his own [1908, n.9.]; reprinted in van Heijenoort [1967, p. 191]).

THE PEIRCIAN INFLUENCE ON WHITEHEAD AND RUSSELL

Still, I thought, Russell and Whitehead themselves certainly learned *their* logic from Frege. To check this I turned to Russell's autobiographical writings. The result was frustrating. In *My Philosophical Development* [1959], Russell describes the impact that meeting Peano had upon his logical development. Strangely enough, he does not mention the *quantifier*, which seems so very central from our present point of view, at all.

Peano taught Russell what was a commonplace in the Peirce-Schröder logical community, the difference in logical form between *all men are mortal* and *Socrates is a man*. And it is clear that one of the notations used in *Principia* for a universally quantified conditional--writing the variable of quantification under the sign of the conditional--came from Peano. But the quantifier as such is not something that Russell singled out for discussion (unless there is something in the unpublished *Nachlass* in the Russell Archives in Ontario). Even when Russell discusses his debt to Frege (in a peculiar way: Russell is unstinting in his praise of Frege's genius, but claims to have thought of the definition of number quite independently), he does not mention the quantifier. *Principia* is no more help on this score, although there is an indication in it that most of the specific notations were invented by Whitehead rather than Russell.

Since I have mentioned Peano, I should remark that he was not only well acquainted with Peirce-Schröder logic, but he had actually corresponded with Peirce. As far as I know, no one has suggested that his logical contributions are independent of the main-line "Boolean" tradition.

In desperation, I looked at Whitehead's *Universal Algebra*. This is a work squarely in the tradition to which Boole, Schröder, and Peirce belonged, the tradition that treated general algebra and logic as virtually one subject. And here, before Whitehead worked with Russell, there is no mention of Frege but there is a citation of "suggestive papers," by Peirce's students O. H. Mitchell and Christine Ladd-Franklin [Whitehead 1898, p. 116]. The topic, of course, is the quantifier.

In sum, I do not know where Russell learned quantification; but Whitehead certainly came to his knowledge of it through "Peirce and his students." On the other hand, the axioms in *Principia* are almost certainly derived from Frege's *Begriffsschrift*; Peirce gave no system of axioms for first-order logic, although his "existential graphs" are a complete proof procedure for first-order logic (an early form of natural deduction).

NOT TO GO OVERBOARD

I have, if anything, *minimized* Frege's contribution and played up the Boolean contribution for reasons which I have explained. But to leave matters here would be as unjust to Frege and to a third tradition, the Hilbert tradition (proof theory), as Quine's unfortunate remark was to the Boolean tradition.

Frege's work is sometimes pooh-poohed today (I mean Frege's logical achievement; Frege's stock as a *philosopher* has never

been higher), though not, of course, by Quine. It is conceded that Frege was far more rigorous and, in particular, far more consistently free of use-mention confusions than other logicians; but such domestic virtue is no longer felt to be impressive. The central charge laid against his work (and Whitehead and Russell's) is that what they called logic is not logic but "set theory," and that reducing arithmetic to set theory is a bad idea.

This raises philosophical issues far too broad for this paper. But let me just make two comments on this: (1) Where to draw the line between logic and set theory (or predicate theory) is *not* an easy question. The statement that a syllogism is *valid*, for example, is a statement of *second-order* logic. (*Barbara* is valid just in case $(F)(G)(H)((Fx \supset Gx) \cdot (Gx \supset Hx) \supset (Fx \supset Hx))$, for example.) If second-order logic is "set theory," then most of traditional logic thus becomes "set theory." (2) The full intuitive principle of mathematical induction is definitely second order in anybody's view. Thus there is a higher-order element in arithmetic whether or not one chooses to "identify numbers with sets." (Just as Frege realized.)

But, philosophical questions aside, Frege certainly undertook one of the most ambitious logical investigations in all history. Its enormous sweep made it (after its repair by Whitehead and Russell, and its translation into a notation resembling Peirce's) a great stimulus to all future work in the field. The Hilbert school certainly put it in the center of their proof theoretic investigations: Gödel's most famous paper [1931], after all, bears the title "*On Principia Mathematica and Related Systems, I.*" That all its achievements could be imitated successfully by the Cantorians (Zermelo and von Neumann) does not take away either its priority or its influence. If Peirce and Schröder were the cutting edge of the logical world prior to *Principia Mathematica* (or a cutting edge--the Hilbert school was already under way), after the appearance of *Principia* their work lost its importance--or lost it except for one important thing: its influence on Hilbert, who followed Peirce in separating first-order logic off from the higher system for metamathematical study.

Principia in turn was to lose its "cutting edge" position when interest shifted from the construction of systems (and the derivation of mathematics within them) to the metamathematical study of properties of systems. Nothing remains forever the "cutting edge" in a healthy science. But a fair-minded statement of the historic importance of the different schools of work, a statement that does justice to each without slighting the others, should not be impossible. Such a statement was given by Hilbert and Ackermann:

The first clear idea of a mathematical logic was formulated by Leibniz. The first results were obtained by A. de Morgan (1806-1876) and G. Boole (1815-1864). The entire later development goes back to Boole. Among his successors, W. S. Jevons (1835-1882) and especially C. S. Peirce (1839-1914) enriched the young science. Ernst Schröder systematically organized and supplemented the various results of his predecessors in his *Vorlesungen über die Algebra der Logik* (1890-1895), which represents a certain completion of the series of developments proceeding from Boole.

In part independently of the development of the Boole-Schröder algebra, symbolic logic received a new impetus from the need of mathematics for an exact foundation and strict axiomatic treatment. G. Frege published his *Begriffsschrift* in 1879 and his *Grundgesetze der Arithmetik* in 1893-1903. G. Peano and his co-workers began in 1894 the publication of the *Formulaire des Mathématiques*, in which all the mathematical disciplines were to be presented in terms of the logical calculus. A high point of this development is the appearance of the *Principia Mathematica* (1910-1913) by A. N. Whitehead and B. Russell. Most recently Hilbert, in a series of papers and university lectures, has used the logical calculus to find a new way of building up mathematics which makes it possible to recognize the consistency of the postulates adopted. The first comprehensive account of these researches has appeared in the *Grundlagen der Mathematik* (1934-1939), by D. Hilbert and P. Bernays. [Hilbert and Ackerman 1938, 1-2]

If Quine had produced this statement in his preface, I should not have had a topic for this conference.

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